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CONICAL BODIES OF GIVEN LENGTH AND VOLUME HAVING MAXIMUM LIFT-TO-DRAG RATIO AT HYPERSONIC SPEEDS

Variational Methods

by Ho-Yi Huang

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CONICAL BODIES OF GIVEN LENGTH AND VOLUME HAVING MAXIMUM LIFT-TO-DRAG RATIO AT HYPERSONIC SPEEDS

Variational Methods[1]

By Ho-Yi Huang[2]

SUMMARY

An investigation of the lift-to-drag ratio attainable by a slender, conical body flying at hypersonic speeds is presented under the assumptions that the pressure distribution is modified Newtonian and the surface-averaged friction coefficient is constant. The length and the volume are given, and the values of the free-stream dynamic pressure, the factor modifying the Newtonian pressure distribution, and the surface-averaged friction coefficient are known a priori. The indirect methods of the calculus of variations are employed and numerical solutions are found using an IBM 7040 computer.

It is found that a unique solution exists for each value of the dimensionless parameter $S_* = (3V/\ell^3)\sqrt[3]{(n/C_f)^2}$, where V is the volume, ℓ the length, n the factor modifying the Newtonian pressure distribution, and C_f the surface-averaged friction coefficient. As S_* increases, the maximum lift-to-drag ratio increases, tending to the limiting value $E = 0.529\sqrt[3]{(n/C_f)}$ when $S_* \to \infty$. The optimum configuration is body-like for relatively small values of the

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parameter S_* and wing-like for relatively large values of the parameter S_*.

1. INTRODUCTION

In Refs. 1 and 2, the lift-to-drag ratio obtainable by a slender, homothetic body at hypersonic speeds was studied under the assumptions that the pressure coefficient is modified Newtonian and the surface-averaged friction coefficient is constant. Attention was focused on conical configurations whose length is given, whose volume is unconstrained, and whose cross-sectional elongation ratio is prescribed. Direct methods were employed in Ref. 1 and indirect methods in Ref. 2.

In Ref. 3, a problem complementary to that investigated in Refs. 1 and 2 was studied: that of conical configurations whose length and volume are given, while the cross-sectional elongation ratio is free. Direct methods were used, various types of transversal contours were analyzed, and the maximum lift-to-drag ratio was determined for each transversal contour.

In this report, we study the problem of Ref. 3 once more. However, no limitation is imposed on the geometry of the cross section, and the indirect methods of the calculus of variations are employed to find the transversal contour which maximizes the lift-to-drag ratio. We employ the following hypotheses: (a) the body is conical, (b) the body is slender in the longitudinal sense, (c) a plane of symmetry exists between the left-hand and right-hand sides of the body, (d) the base plane is perpendicular to the plane of symmetry, (e) the free-stream velocity is contained in the plane of symmetry and is perpendicular to the base plane, (f) the pressure distribution is modified Newtonian, that is, the pressure coefficient is proportional to the cosine squared of the angle between the local normal to the body and the undisturbed flow direction, (g) the surface-averaged friction coefficient is constant, (h) the base drag coefficient is zero, and (i) the contribution of the tangential forces to the lift is negligible with respect to the contribution of the normal forces.

2. LIFT-TO-DRAG RATIO

Consider the Cartesian coordinate system Oxyz and the cylindrical coordinate system $Oxr\theta$ shown in Fig. 1. For the Cartesian system, the origin O is at the apex of the body, the x-axis is parallel to the free-stream velocity and positive toward the base, the z-axis is contained in the plane of symmetry and positive downward, and the y-axis is oriented in such a way that the xyz-system is right-handed. For the cylindrical system, r is the distance of any point from the x-axis, and θ measures the angular position of the vector $\vec{r}$ with respect to the xy-plane.

In the cylindrical coordinate system, the geometry of a conical body can be written in the form

$$r = (x/\ell)R \tag{1}$$

where ℓ denotes the length of the body and $R = R(\theta)$ is a function describing the base contour. Therefore, if the hypotheses of the introduction are employed, the drag D and the lift L per unit free-stream dynamic pressure q_∞ can be written as (Ref. 3)

$$\begin{aligned} D/q_\infty &= \int_{-\pi/2}^{\pi/2} [2nR^6/\ell^2(R^2 + \dot{R}^2) + C_f \ell \sqrt{(R^2 + \dot{R}^2)}]\, d\theta \\ L/q_\infty &= \int_{-\pi/2}^{\pi/2} [2nR^4/\ell(R^2 + \dot{R}^2)]\,(R \sin\theta - \dot{R}\cos\theta)\, d\theta \end{aligned} \tag{2}$$

where C_f denotes the surface-averaged friction coefficient, assumed constant, n a factor modifying the Newtonian pressure law[3], and $\dot{R}$ the derivative $dR/d\theta$.

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[3] Under the slender body approximation, the pressure coefficient for a conical body is given by $C_p = 2nR^4/\ell^2(R^2 + \dot{R}^2)$.

If one introduces the constant

$$f = \sqrt[3]{(C_f/n)} \tag{3}$$

and the dimensionless quantities

$$\rho = R/\ell f \quad , \quad D_* = D/nq_\infty \ell^2 f^4 \quad , \quad L_* = L/nq_\infty \ell^2 f^3 \tag{4}$$

the previous relations become

$$\begin{aligned} D_* &= \int_{-\pi/2}^{\pi/2} [2\rho^6/(\rho^2 + \dot{\rho}^2) + \sqrt{(\rho^2 + \dot{\rho}^2)}]\, d\theta \\ L_* &= \int_{-\pi/2}^{\pi/2} [2\rho^4/(\rho^2 + \dot{\rho}^2)](\rho \sin\theta - \dot{\rho}\cos\theta)\, d\theta \end{aligned} \tag{5}$$

After the lift-to-drag ratio E and the modified lift-to-drag ratio E_* are defined as

$$E = L/D \quad , \quad E_* = Ef \tag{6}$$

one obtains the relationship

$$E_* = L_*/D_* \tag{7}$$

Clearly, the modified lift-to-drag ratio is uniquely determined once the dimensionless base radius function $\rho(\theta)$ is prescribed.

3. BASE AREA AND VOLUME

Regardless of whether or not the body is slender, the base area and the volume of a conical body are given by (Ref. 3)

$$S = 3V/\ell = \int_{-\pi/2}^{\pi/2} R^2 d\theta \tag{8}$$

Therefore, assigning the length and the base area is equivalent to specifying the volume; conversely, assigning the length and the volume is equivalent to specifying the base area. After the dimensionless area S_* and the dimensionless volume V_* are defined as

$$S_* = S/\ell^2 f^2 \quad , \quad V_* = V/\ell^3 f^2 \tag{9}$$

one concludes that

$$S_* = 3V_* = \int_{-\pi/2}^{\pi/2} \rho^2 d\theta \tag{10}$$

4. OPTIMUM TRANSVERSAL CONTOUR PROBLEM

We assume that the length ℓ and the volume V (and, hence, the base area S) are given. We also assume that the free-stream conditions are prescribed and that the quantities n and C_f are known *a priori*. Therefore, the dimensionless area S_* is known *a priori*. For each value of S_*, there exist an infinite number of cross-sectional shapes $\rho(\theta)$ which satisfy the isoperimetric constraint (10). Among these, we want to find the one that maximizes the modified lift-to-drag ratio (7), where L_* and D_* are given by Eqs. (5).

We restrict our attention to the class of cross-sectional shapes described in Fig. 2. Here, O denotes the projection of the apex of the conical body on the base section, I is the maximum width point, and F is the point at which $\theta = \pi/2$. These cross sections include an upper contour OI which is rectilinear and a lower contour IF whose equation $\rho = \rho(\theta)$ is to be found by variational methods; hence, these cross sections are described by

$$\begin{aligned} \rho &= 0 \quad , & -\pi/2 \le \theta \le \theta_i \\ \dot{\rho} &= \infty \quad , & \theta = \theta_i \\ \rho &= \rho(\theta) \quad , & \theta_i \le \theta \le \pi/2 \end{aligned} \tag{11}$$

where θ_i is the angular coordinate of the maximum width point. Note that the upper contour has zero lift, zero pressure drag, and positive friction drag and the lower contour has positive lift, pressure drag, and friction drag. Therefore, the dimensionless drag, lift, area, and volume become

$$
\begin{aligned}
D_* &= \rho_i + \int_{\theta_i}^{\pi/2} [2\rho^6/(\rho^2 + \dot{\rho}^2) + \sqrt{(\rho^2 + \dot{\rho}^2)}]\, d\theta \\
L_* &= \int_{\theta_i}^{\pi/2} [2\rho^4/(\rho^2 + \dot{\rho}^2)](\rho \sin \theta - \dot{\rho} \cos \theta)\, d\theta \qquad (12) \\
S_* &= 3V_* = \int_{\theta_i}^{\pi/2} \rho^2 d\theta
\end{aligned}
$$

where ρ_i denotes the radius at the maximum width point. In the light of these equations, the problem of optimizing the transversal contour for given length and volume (and, hence, for given length and base area) is to find, among the functions $\rho = \rho(\theta)$ which satisfy the isoperimetric constraint (12-3), the one that maximizes the modified lift-to-drag ratio (7), where D_* and L_* are given by Eqs. (12-1) and (12-2).

5. NECESSARY CONDITIONS

In accordance with the treatment of Refs. 4 and 5, the necessary conditions for the problem stated in the previous section are identical with those characterizing the functional

$$I = \int_{\theta_i}^{\theta_f} F(\theta, \rho, \dot{\rho}, \lambda, \mu) d\theta + G(\rho_i, \mu) \tag{13}$$

where F and G denote the functions

$$\begin{aligned} F &= [2\rho^4/(\rho^2 + \dot{\rho}^2)](\rho \sin\theta - \dot{\rho}\cos\theta) - \mu[2\rho^6/(\rho^2 + \dot{\rho}^2) + \sqrt{(\rho^2 + \dot{\rho}^2)}] - \lambda\rho^2 \\ G &= -\mu\rho_i \end{aligned} \tag{14}$$

and where the subscripts i and f stand for initial and final points, respectively.

In Eqs. (14), λ and μ are constant Lagrange multipliers. The former must be determined so that the isoperimetric constraint (12-3) is satisfied and the latter so that

$$\mu = E_* \tag{15}$$

where E_* is the unknown maximum value of the modified lift-to-drag ratio. The fundamental function (14-1) is characterized by the first partial derivatives

$$\begin{aligned} F_\rho &= [2\rho^3/(\rho^2 + \dot{\rho}^2)^2][(3\rho^2 + 5\dot{\rho}^2)\rho \sin\theta - 2(\rho^2 + 2\dot{\rho}^2)\dot{\rho}\cos\theta - 2\mu\rho^2(2\rho^2 + 3\dot{\rho}^2)] \\ &\quad - \mu\rho/\sqrt{(\rho^2 + \dot{\rho}^2)} - 2\lambda\rho \\ F_{\dot{\rho}} &= [2\rho^4/(\rho^2 + \dot{\rho}^2)^2][-2\rho\dot{\rho}\sin\theta - (\rho^2 - \dot{\rho}^2)\cos\theta + 2\mu\rho^2\dot{\rho}] - \mu\dot{\rho}/\sqrt{(\rho^2 + \dot{\rho}^2)} \end{aligned} \tag{16}$$

and the second partial derivatives

$$F_{\dot{\rho}\theta} = [2\rho^4/(\rho^2 + \dot{\rho}^2)^2][(\rho^2 - \dot{\rho}^2) \sin \theta - 2\rho\dot{\rho} \cos \theta]$$

$$F_{\dot{\rho}\rho} = [4\rho^3/(\rho^2 + \dot{\rho}^2)^3][-(\rho^2 + 5\dot{\rho}^2)\rho\dot{\rho} \sin \theta - (\rho^4 + 3\rho^2\dot{\rho}^2 - 2\dot{\rho}^4) \cos \theta$$

$$+ 2\mu\rho^2\dot{\rho}(\rho^2 + 3\dot{\rho}^2)] + \mu\rho\dot{\rho}/(\rho^2 + \dot{\rho}^2)^{3/2} \tag{17}$$

$$F_{\dot{\rho}\dot{\rho}} = [4\rho^4/(\rho^2 + \dot{\rho}^2)^3][-(\rho^2 - 3\dot{\rho}^2)\rho \sin \theta + (3\rho^2 - \dot{\rho}^2)\dot{\rho} \cos \theta + \mu(\rho^2 - 3\dot{\rho}^2)\rho^2]$$

$$- \mu\rho^2/(\rho^2 + \dot{\rho}^2)^{3/2}$$

From calculus of variations (see, for instance, Chapter 2 of Ref. 6), it is known that the extremal arc must be a solution of the Euler equation

$$F_{\dot{\rho}\dot{\rho}}\ddot{\rho} + F_{\dot{\rho}\rho}\dot{\rho} + F_{\dot{\rho}\theta} - F_{\rho} = 0 \tag{18}$$

This solution must satisfy the initial conditions

$$(F + \mu\dot{\rho})_i = 0 \quad , \quad (F_{\dot{\rho}} + \mu)_i = 0 \tag{19}$$

and the final conditions

$$\theta_f = \pi/2 \quad , \quad (F_{\dot{\rho}})_f = 0 \tag{20}$$

Note that Eqs. (19-1), (19-2), and (20-2) are natural boundary conditions which arise from the transversality conditions. They express the optimal choice of the initial angle θ_i, the initial radius ρ_i, and the final radius ρ_f, respectively.

Once a solution satisfying the Euler equation (18) and the boundary conditions (19) and (20) is found, one has to verify that it yields a maximum for the functional (13). In this connection, the Weierstrass condition is of considerable assistance. It states that the excess function

$$e = \Delta F - F_{\dot{\rho}} \Delta\dot{\rho} \tag{21}$$

must be negative at every point of the maximal arc. In Eq. (21), the symbols ΔF and $\Delta\dot{\rho}$ are defined as

$$\begin{aligned} \Delta F &= F(\theta, \rho, \dot{\rho}_*, \lambda, \mu) - F(\theta, \rho, \dot{\rho}, \lambda, \mu) \\ \Delta\dot{\rho} &= \dot{\rho}_* - \dot{\rho} \end{aligned} \tag{22}$$

where the unstarred quantities refer to the extremal arc and the starred quantities to the comparison arc. On account of Eqs. (14-1), (16-2), and (17-3), the Weierstrass condition can be rewritten as

$$(\rho^2 + \dot{\rho}^2)F_{\dot{\rho}\dot{\rho}}/2 - F_{\dot{\rho}}\Delta\dot{\rho} + \mu H/\sqrt{(\rho^2 + \dot{\rho}^2)} \leq 0 \tag{23}$$

where

$$H = \rho^2/2 - (\rho^2 + \dot{\rho}^2)K^2/(K + 1) + \dot{\rho}[2\dot{\rho}K^2 - (2K + 1)\Delta\dot{\rho}]/(K + 1)^2 \tag{24}$$

and

$$K = \sqrt{[1 + 2\dot{\rho}\Delta\dot{\rho}/(\rho^2 + \dot{\rho}^2) + (\Delta\dot{\rho})^2/(\rho^2 + \dot{\rho}^2)]} \tag{25}$$

6. SOLUTION PROCESS

The Euler equation (18) is a second-order, nonlinear differential equation. In the light of Eqs. (16) and (17), it is clear that finding analytical solutions is rather difficult. For this reason, the integration of the differential equation (18) has been carried out numerically with an IBM 7040 computer. We describe the solution process below.

Since the final conditions (20) are simpler than the initial conditions (19), we integrate the Euler equation (18) backward. To start the integration, the quantities θ_f, ρ_f, $\dot{\rho}_f$ and the constants λ, μ are needed. Among these five quantities, two relations exist. First, θ_f is given by Eq. (20-1). Next, $\dot{\rho}_f$ can be determined from Eq. (20-2) which, in the light of Eq. (16-2), yields either of the following relationships:

$$\dot{\rho}_f = 0 \quad , \quad \dot{\rho}_f = \pm \sqrt{\{[4\rho_f^5(\rho_f - 1/\mu)]^{2/3} - \rho_f^2\}} \tag{26}$$

Therefore, we chose ρ_f, λ, and μ as the three parameters of our problem.

From the variational approach to the problem where the volume is free (Ref. 2), we know that the values of the modified lift-to-drag ratio are in the range

$$0 \leq E_* \leq 0.529 \tag{27}$$

Because of Eq. (15), we surmise that the values of the Lagrange multiplier μ are in the range

$$0 \leq \mu \leq 0.529 \tag{28}$$

This being the case, it is convenient to fix a value of μ within this range and systematically vary ρ_f and λ up to the moment when the relationships (15) and (19) are satisfied. Once a solution $\rho = \rho(\theta)$ has been found for a given μ, the corresponding dimensionless area S_* is obtained a posteriori from Eq. (12-3).

The first step of the analysis is to choose a value for μ and guess a pair of values for ρ_f and λ. The value of $\dot{\rho}$ is determined by either Eq. (26-1) or Eq. (26-2). With this information, we integrate the Euler equation (18) backward. At every station θ, the quantities ρ and $\dot{\rho}$ are given by functional relationships of the form

$$\rho = f_1(\theta, \lambda, \rho_f, \mu) \quad , \quad \dot{\rho} = f_2(\theta, \lambda, \rho_f, \mu) \tag{29}$$

Also, the left-hand sides of Eqs. (19) have the form

$$F + \mu\dot{\rho} = f_3(\theta, \lambda, \rho_f, \mu) \quad , \quad F_{\dot{\rho}} + \mu = f_4(\theta, \lambda, \rho_f, \mu) \tag{30}$$

The vanishing of either f_3 or f_4 can be chosen to be the stopping condition of the integration process.

The second step is to maintain constant values for μ and ρ_f and systematically vary λ to ensure that the left-hand sides of Eqs. (19) vanish simultaneously. This occurs when

$$f_3(\theta_i, \lambda, \rho_f, \mu) = 0 \quad , \quad f_4(\theta_i, \lambda, \rho_f, \mu) = 0 \tag{31}$$

Computationally speaking, this step can be carried out providing ρ_f is in a proper range and leads to solutions having the form

$$\theta_i = f_5(\rho_f, \mu) \quad , \quad \lambda = f_6(\rho_f, \mu) \tag{32}$$

Once the values of λ and θ_i have been determined, a contour satisfying the Euler equation (18) and the boundary conditions (19) and (20) has been found. The associated modified lift-to-drag ratio E_*, obtained from Eqs. (12-1) and (12-2) in combination with the definition (7), has the form

$$E_* = f_7(\rho_f, \mu) \tag{33}$$

Generally speaking, the modified lift-to-drag ratio given by Eq. (33) does not satisfy Eq. (15). Therefore, the next step is to vary ρ_f and repeat the previous procedure up to the moment when the relation

$$\mu = f_7(\rho_f, \mu) \tag{34}$$

is satisfied for the assigned value of μ. Clearly, Eq. (34) admits a solution of the form

$$\rho_f = f_8(\mu) \tag{35}$$

After Eqs. (15), (19),and (20) are satisfied, the area constraint S_* can be determined a posteriori by integrating Eq. (12-3).

In order to ensure that the solutions obtained are locally optimal, the Weierstrass condition must be verified at every step of the integration process. The numerical analysis shows that Ineq. (23) is satisfied everywhere providing the final condition (26-1) is employed, while it is violated along some portion of the extremal arc if the final condition (26-2) is used. This being the case, we conclude that Eq. (26-1) is the correct final condition for the present problem.

The numerical results are presented in Figs. 3 through 8 where the initial angle θ_i, the initial radius ρ_i, the final radius ρ_f, the Lagrange multiplier λ, the modified lift-to-drag ratio E_*, and the cross-sectional elongation ratio

$$\alpha = (\rho_i/\rho_f) \cos \theta_i \tag{36}$$

are plotted versus the dimensionless area S_*. Shown in Figs. 9 and 10 are some optimum contours for several values of the dimensionless area S_*. The characteristics of these contours are summarized in Table 1.

Table 1. Characteristics of the Optimum Contours

S_*	θ_i (degrees)	ρ_i	ρ_f	λ	E_*	α
0.540	64.3	1.133	1.080	0.735	0.400	0.455
0.814	55.0	1.245	1.112	0.514	0.450	0.643
1.070	48.0	1.370	1.135	0.384	0.475	0.807
1.589	37.7	1.669	1.168	0.230	0.500	1.131
2.031	31.5	1.952	1.185	0.161	0.510	1.405
3.033	22.8	2.654	1.211	0.080	0.520	2.020

7. DISCUSSION AND CONCLUSIONS

In the previous sections, the optimization of the lift-to-drag ratio of a slender, conical body flying at hypersonic speeds is presented under the assumptions that the pressure distribution is modified Newtonian and the surface-averaged friction coefficient is constant. The length and the volume are given, and the values of the free-stream dynamic pressure, the factor modifying the Newtonian pressure distribution, and the surface-averaged friction coefficient are known a priori. The indirect methods of the calculus of variations are employed,and numerical solutions are determined using an IBM 7040 computer. It is found that a one-parameter family of extremal solutions exists, the parameter being the dimensionless area S_*. It is also found that the geometry of the optimum contour and the maximum value of the modified lift-to-drag ratio E_* are uniquely related to the dimensionless area S_*.

The results are summarized in Figs. 3 through 8, and some optimum contours are shown in Figs. 9 and 10. It is seen that, as the dimensionless area S_* increases, the initial angle θ_i decreases, while the initial radius ρ_i, the final radius ρ_f, the elongation ratio α, and the modified lift-to-drag ratio E_* increase. For relatively small values of S_*, the cross sections are body-like, have steeper upper contours, and are less efficient aerodynamically. For relatively large values of S_*, the cross sections are wing-like, have flatter upper contours, and are more efficient aerodynamically. As $S_* \to \infty$, the cross section becomes flat-topped and the modified lift-to-drag ratio approaches the limiting value $E_* = 0.529$.

It is of interest to compare the solutions of this report with those obtained in Ref. 3 by direct methods. As expected, for any dimensionless area in the range $0 \le S_* \le \infty$, the lift-to-drag ratios of the variational solutions are higher than those of the shapes investigated in Ref. 3. However, the relative differences are negligible if the extremal contours are compared with the most efficient contours of Ref. 3, namely, the diamond shape and the lenticular shape.

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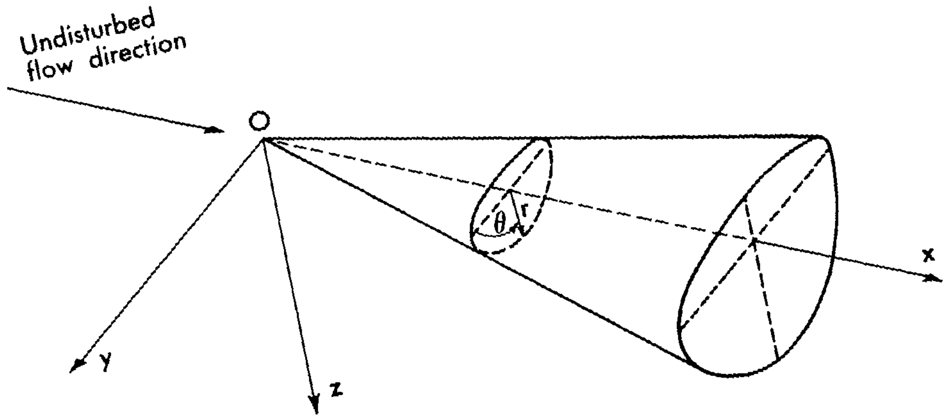

Fig. 1 Coordinate system.

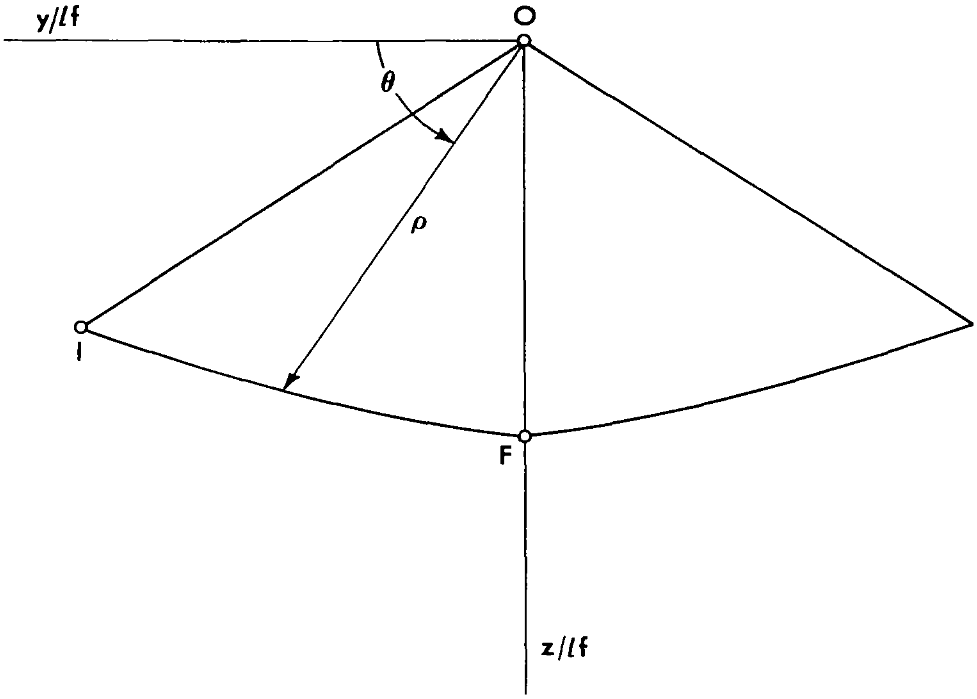

Fig. 2 Cross sections analyzed.

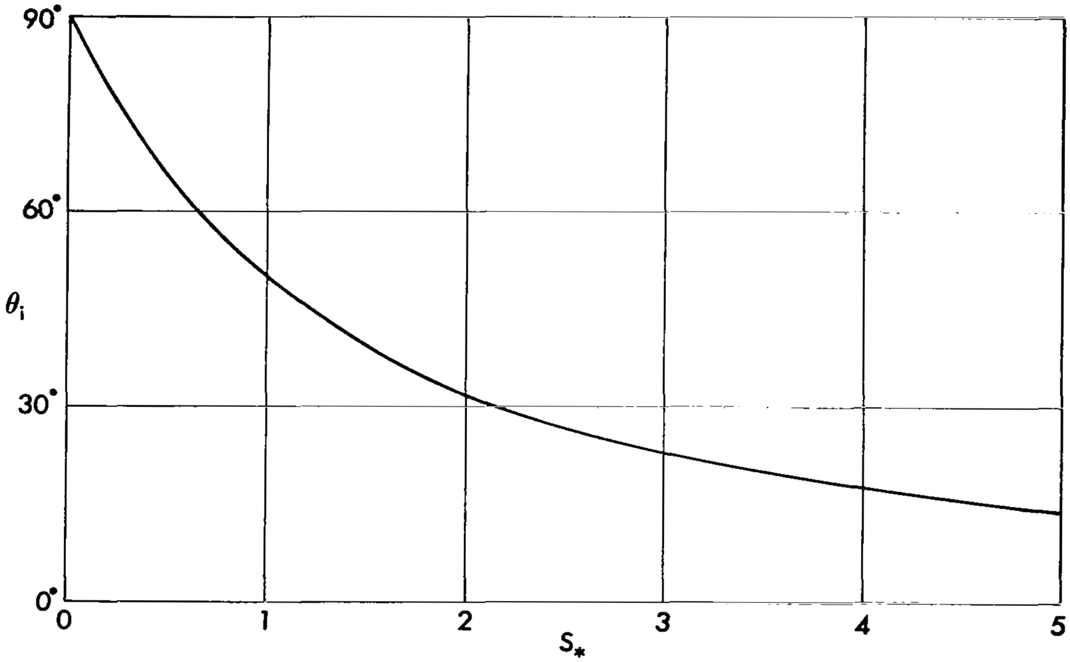

Fig. 3 Initial angle.

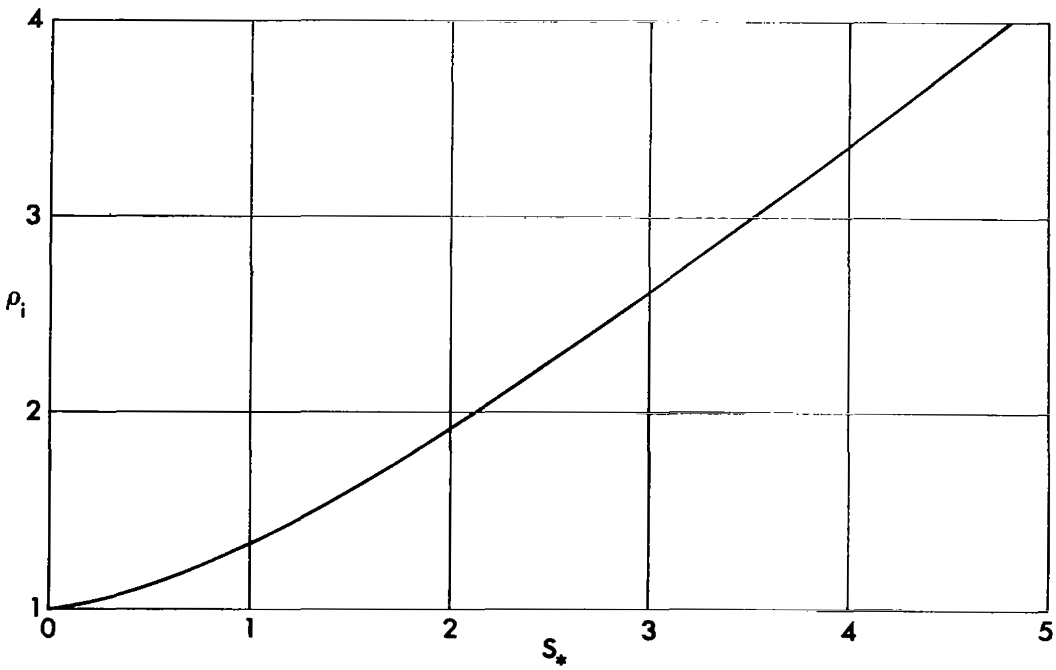

Fig. 4 Initial radius.

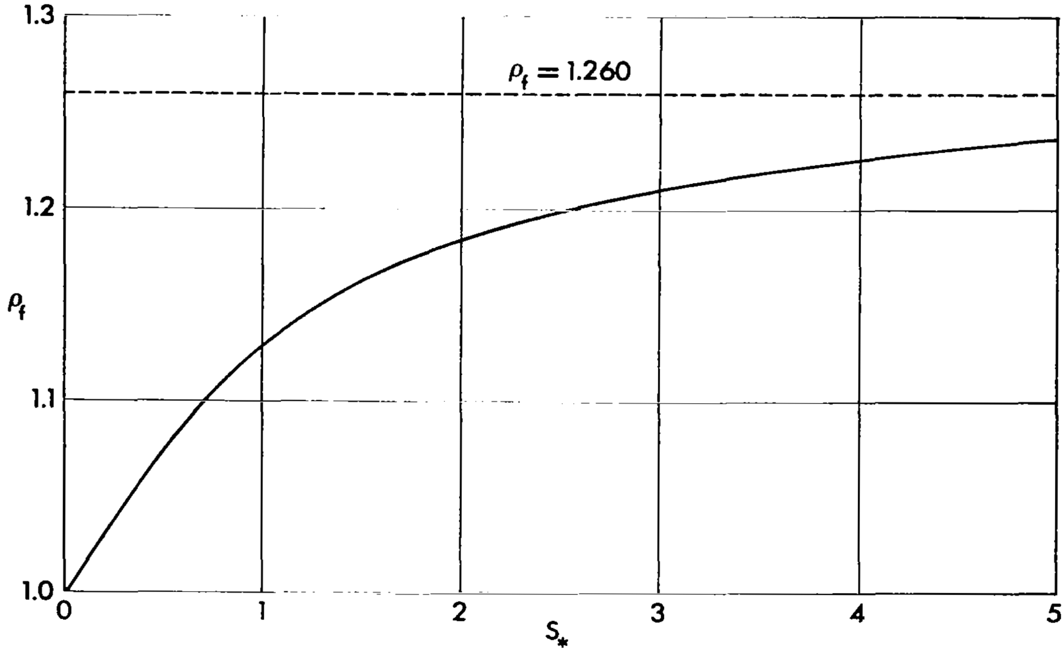

Fig. 5 Final radius.

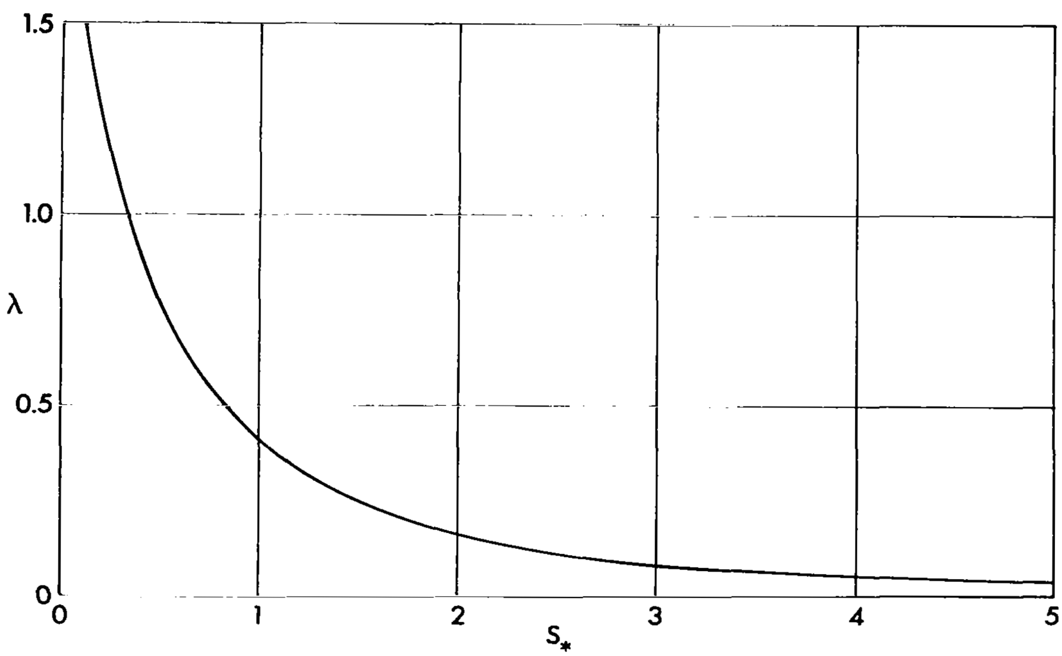

Fig. 6 Lagrange multiplier.

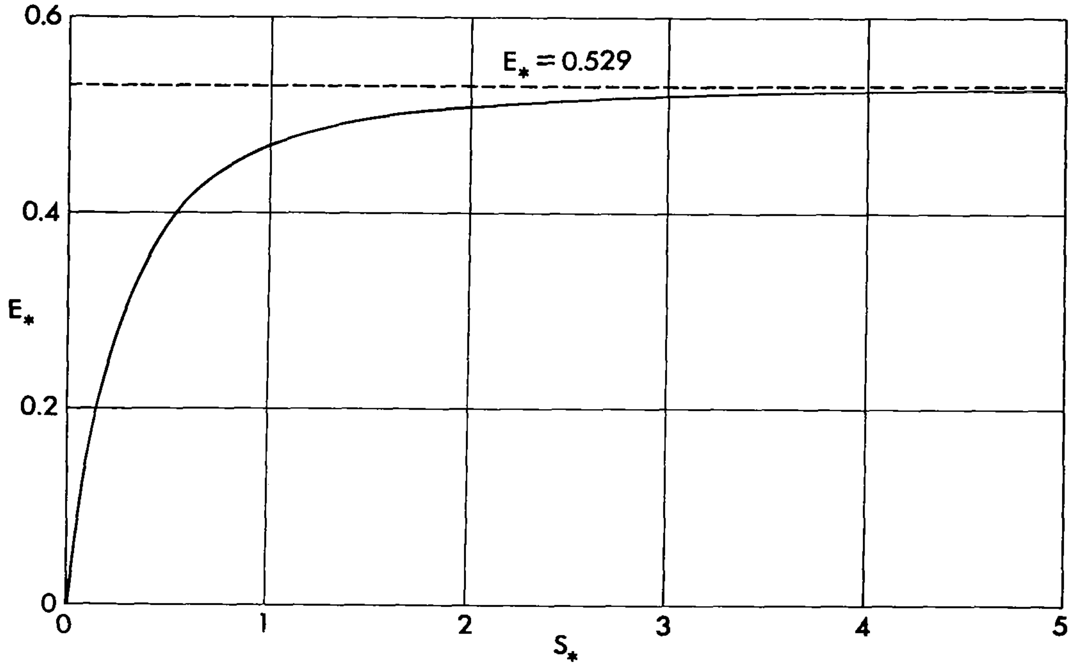

Fig. 7 Lift-to-drag ratio.

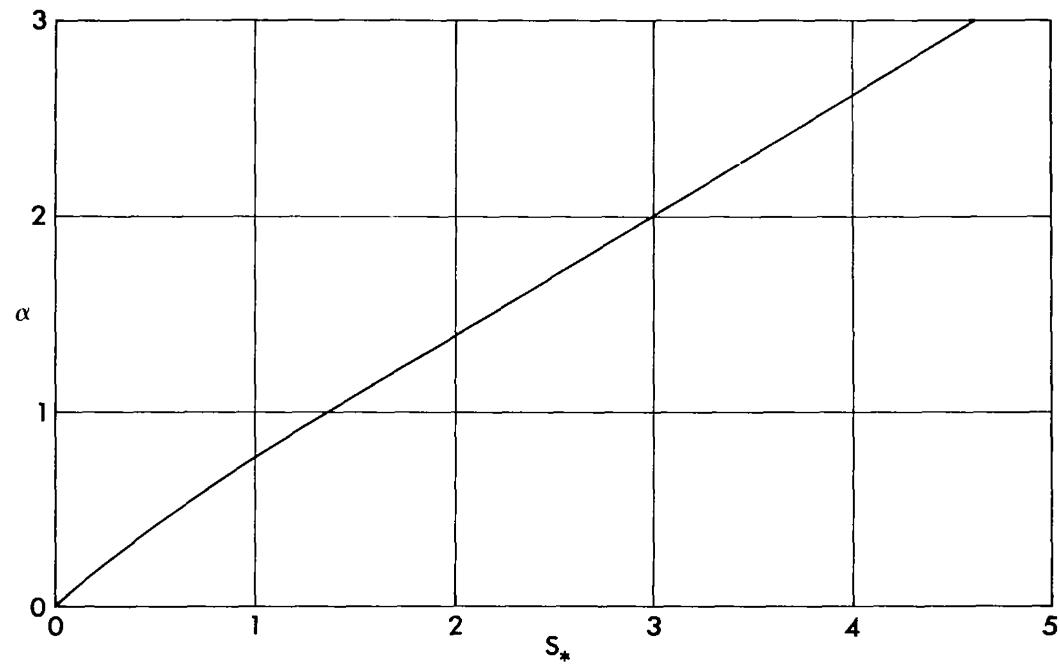

Fig. 8 Elongation ratio.

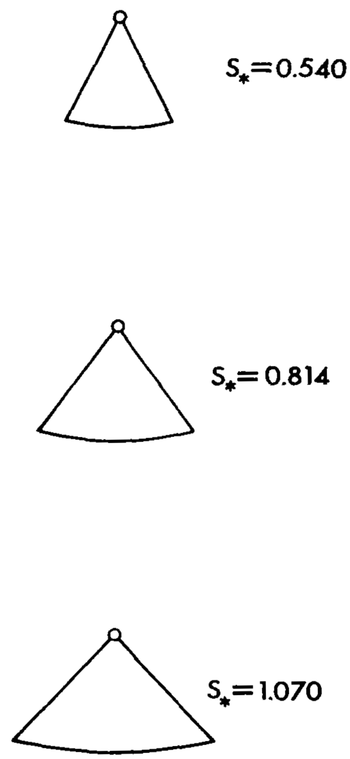

Fig. 9 Optimum contours.

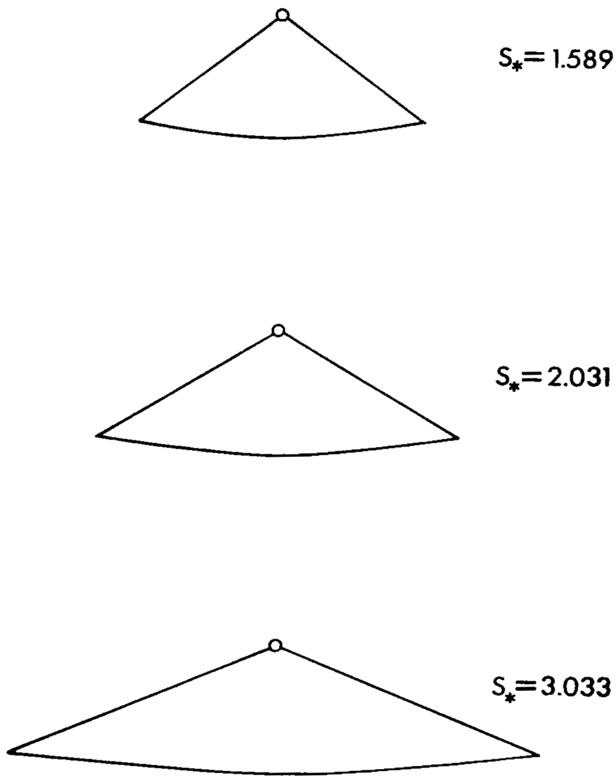

Fig. 10 Optimum contours.

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